19.

De transformatione integralis duplicis indefiniti

$$\int \frac{\partial\varphi\,\partial\psi}{A+B\cos\varphi+C\sin\varphi+(A'+B'\cos\varphi+C'\sin\varphi)\cos\psi+(A''+B''\cos\varphi+C''\sin\varphi)\sin\psi}$$

in formam simpliciorem

$$\int \frac{\partial\eta\,\partial\vartheta}{G-G'\cos\eta\cos\vartheta-G''\sin\eta\sin\vartheta}.$$

(Auct. *C. G. J. Jacobi*, prof. math. Regiom.)

Introductio.

1.

Facile probatur, integrale huiusmodi

$$\int \frac{\partial\varphi}{\sqrt{(a+b\cos\varphi+c\sin\varphi+d\cos\varphi^2+e\cos\varphi\sin\varphi+f\sin\varphi^2)}},$$

in quo expressio, quae sub radicali invenitur, functio est rationalis integra secundi ordinis ipsorum $\cos\varphi$, $\sin\varphi$, casu quo expressio illa pro omnibus anguli φ valoribus realibus valorem positivum servat, per substitutionem realem formae

$$\operatorname{tang}\tfrac{1}{2}\varphi = \frac{m+n\operatorname{tang}\frac{1}{2}\eta}{1+p\operatorname{tang}\frac{1}{2}\eta},$$

ad hoc simplicioris formae integrale revocari posse

$$\frac{1}{M}\int\frac{\partial\eta}{\sqrt{(1-k^2\sin\eta^2)}},$$

in quo insuper $k^2<1$: qua forma hodie integralia elliptica exhiberi solent.

Ponatur enim

$$\operatorname{tang}\tfrac{1}{2}\varphi = x,$$

integrale illud

$$\int \frac{\partial\varphi}{\sqrt{(a+b\cos\varphi+c\sin\varphi+d\cos\varphi^2+e\cos\varphi\sin\varphi+f\sin\varphi^2)}},$$

abire videmus in integrale sequentis formae:

$$\int\frac{\partial x}{\sqrt{(g+hx+ix^2+kx^3+lx^4)}},$$

in quo expressio sub radicali dignitates omnes ipsius x continet integras positivas usque ad quartam; cuiusmodi integrale Eulerus olim docuit, adhibita substitutione

$$x = \frac{m+ny}{1+py},$$

in simplicius transformari posse hoc

$$\int \frac{\partial y}{\sqrt{(q+ry^2+sy^4)}},$$

in quo sub radicali impares dignitates variabilis non inveniuntur. Substitutionem autem adhibitam

$$x = \frac{m+ny}{1+py}$$

Cl. Legendre demonstravit omnibus casibus realem accipi posse, atque integralia ita reducta facillime revocari ad dictam formam

$$\frac{1}{M}\int \frac{\partial \eta}{\sqrt{(1-k^2 \sin \eta^2)}},$$

idque variis modis pro indole ipsarum q, r, s. E quibus modis est substitutio

$$y = \sqrt[4]{\left(\frac{q}{s}\right)} \operatorname{tang} \tfrac{1}{2}\eta,$$

qua adhibita prodit:

$$\int \frac{\partial y}{\sqrt{(q+ry^2+sy^4)}} = \mp \frac{1}{2\sqrt[4]{(qs)}} \cdot \int \frac{\partial \eta}{\sqrt{\left(1-\frac{2\sqrt{(qs)}-r}{4\sqrt{(qs)}} \sin \eta^2\right)}},$$

quod integrale forma assignata gaudet. Junctis substitutionibus, quibus integrale propositum in formam illam transformatum est, invenimus, siquidem loco $n\sqrt[4]{\left(\frac{q}{s}\right)}$, $p\sqrt[4]{\left(\frac{q}{s}\right)}$ simpliciter n, p scribimus, substitutionem formae propositae:

$$\operatorname{tang} \tfrac{1}{2}\varphi = \frac{m+n \operatorname{tang} \frac{1}{2}\eta}{1+p \operatorname{tang} \frac{1}{2}\eta}.$$

2.

Ut substitutio assignata realis sit, in antecedentibus supponi debet, q, s eodem signo affectas esse. Quod facile probatur locum habere, quoties expressiones sub radicali aut pro nullo aut pro quatuor valoribus realibus variabilis evanescant.

Expressiones enim binae

$$a + b\cos\varphi + c\sin\varphi + d\cos\varphi^2 + e\cos\varphi\sin\varphi + f\sin\varphi^2$$

atque

$$q + ry^2 + sy^4$$

eodem tempore evanescunt, idque pro eodem numero valorum realium et imaginariorum variabilium. Quoties vero q, s signa opposita habent, evanescit haec pro valoribus realibus ipsius y^2 uno positivo, uno negativo; unde valores variabilium y, φ, pro quibus expressiones illae evanescunt, eo casu duo reales, duo imaginarii forent.

Porro facile probatur, altero casu, quo expressio sub radicali pro nullo valore reali variabilis evanescat sive valorem semper positivum servet, modulum integralis elliptici, ad quod integrale propositum revocatur, semper realem unitate minorem effici posse.

Quem in finem observo, substitutionem nostram

$$\operatorname{tang}\tfrac{1}{2}\varphi = \frac{m+n\operatorname{tang}\frac{1}{2}\eta}{1+p\operatorname{tang}\frac{1}{2}\eta}$$

formam non mutare, ubi loco $\operatorname{tang}\frac{1}{2}\eta$ ponatur

$$\frac{1-\operatorname{tang}\frac{1}{2}\eta}{1+\operatorname{tang}\frac{1}{2}\eta},$$

sive loco η ponatur $90^\circ-\eta$. Quo facto integrale reductum abit in

$$\frac{1}{\sqrt{(2\sqrt{(qs)}+r)}}\int\frac{\partial\eta}{\sqrt{\left(1-\frac{r-2\sqrt{(qs)}}{r+2\sqrt{(qs)}}\sin\eta^2\right)}},$$

ita ut quadratum moduli invenias

$$\text{aut}\quad k^2=\frac{2\sqrt{(qs)}-r}{4\sqrt{(qs)}}\quad\text{aut}\quad k^2=\frac{r-2\sqrt{(qs)}}{r+2\sqrt{(qs)}}.$$

Quoties vero expressio sub radicali in integrali proposito valorem semper positivum habet, radices y^2 aequationis quadraticae

$$q+ry^2+sy^4=0$$

aut imaginariae fiunt, aut certe negativae. Casu primo fit $rr<4qs$, ideoque modulus

$$k=\sqrt{\left(\frac{2\sqrt{(qs)}-r}{4\sqrt{(qs)}}\right)}$$

realis unitate minor. Casu secundo fit $rr>4qs$ simulque r positiva, ideoque eo casu modulus

$$k=\sqrt{\left(\frac{r-2\sqrt{(qs)}}{r+2\sqrt{(qs)}}\right)}$$

realis unitate minor. Unde, quod probari debuit, siquidem expressio sub radicali valorem semper positivum habet, per dictam substitutionem omnibus casibus ad integrale ellipticum pervenire licet, cuius modulus realis unitate minor est.

Altero casu, quo denominator integralis pro quatuor valoribus realibus variabilis evanescit, fit $rr>4qs$ simulque r negativa. Quo casu ut modulus realis unitate minor existat, cum ad novas substitutiones confugiendum sit, plerumque eum casum alia ratione tractare praestat.

3.

E relatione, quae inter $\operatorname{tang}\frac{1}{2}\varphi$, $\operatorname{tang}\frac{1}{2}\eta$ obtinet,

$$\operatorname{tang}\tfrac{1}{2}\varphi = \frac{m + n\operatorname{tang}\frac{1}{2}\eta}{1 + p\operatorname{tang}\frac{1}{2}\eta},$$

valores ipsorum $\cos\varphi$, $\sin\varphi$ fluunt huiusmodi:

$$\cos\varphi = \frac{\beta - \beta'\cos\eta - \beta''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

$$\sin\varphi = \frac{\gamma - \gamma'\cos\eta - \gamma''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta}.$$

Quibus in expressionibus inter coëfficientes α, β etc. certae quaedam aequationes conditionales locum habere debent, cum identice fieri debeat:

$$(\alpha - \alpha'\cos\eta - \alpha''\sin\eta)^2 = (\beta - \beta'\cos\eta - \beta''\sin\eta)^2 + (\gamma - \gamma'\cos\eta - \gamma''\sin\eta)^2.$$

Quod etiam inde patet, quod omnes a tribus m, n, p pendent. Vice versa facile probatur, quod infra videbimus, quoties per relationes, quae inter α, β etc. locum habent, identice sit:

$$(\alpha - \alpha'\cos\eta - \alpha''\sin\eta)^2 = (\beta - \beta'\cos\eta - \beta''\sin\eta)^2 + (\gamma - \gamma'\cos\eta - \gamma''\sin\eta)^2,$$

ideoque simul ponere liceat:

$$\cos\varphi = \frac{\beta - \beta'\cos\eta - \beta''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

$$\sin\varphi = \frac{\gamma - \gamma'\cos\eta - \gamma''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

inde etiam relationem illam linearem inter tangentes semiarcuum demanare:

$$\operatorname{tang}\tfrac{1}{2}\varphi = \frac{m + n\operatorname{tang}\frac{1}{2}\eta}{1 + p\operatorname{tang}\frac{1}{2}\eta}.$$

Cui insuper videbimus formam conciliari posse ad calculum idoneam:

$$\operatorname{tang}\tfrac{1}{2}(\varphi' - \varphi)\operatorname{tang}\tfrac{1}{2}(\eta - \eta') = \mu,$$

ubi φ', η', μ' constantes.

4.

Patet ex antecedentibus, substitutionem illam

$$\operatorname{tang}\tfrac{1}{2}\varphi = \frac{m + n\operatorname{tang}\frac{1}{2}\eta}{1 + p\operatorname{tang}\frac{1}{2}\eta},$$

etiam per binas aequationes inter se iunctas repraesentari posse:

$$\cos\varphi = \frac{\beta - \beta'\cos\eta - \beta''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

$$\sin\varphi = \frac{\gamma - \gamma'\cos\eta - \gamma''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

in quibus inter coëfficientes certae relationes locum habent. Quae forma substitutionis non sine elegantia ad transformationem propositam adhibetur, quamvis in locum trium quantitatum m, n, p novem α, β etc. in calculum introducantur, aut certe octo, cum unius ex earum numero valorem pro arbitrio assumere liceat. Id quod licet in casu paullo restrictiore, ad quem tamen generalior facile revocatur, a Cl. Gauss factum est in commentatione *Determinatio attractionis etc.*

Analysin transformationis propositae dictum in modum institutam observari olim (*Diar.* Crell. Vol. II. pag. 228.), prorsus convenire cum problemate algebraico, per substitutiones

$$x = \alpha s + \alpha' s' + \alpha'' s'',$$
$$y = \beta s + \beta' s' + \beta'' s'',$$
$$z = \gamma s + \gamma' s' + \gamma'' s'',$$

quae identice efficiant,

$$xx + yy + zz = ss + s's' + s''s'',$$

simul expressionem

$$ax^2 + bxy + cxz + dy^2 + eyz + fz^2$$

in hanc simpliciorem transformare:

$$GGss + G'G's's' + G''G''s''s'';$$

quod scimus problema investigationem axium principalium ellipsoidarum concernere.

Supponamus enim in problemate illo algebraico

$$xx + yy + zz = ss + s's' + s''s'' = 0;$$

quibus statutis, siquidem $i = \sqrt{-1}$, ponere licet:

$$\frac{y}{x} = -i\cos\varphi, \qquad \frac{s'}{s} = i\cos\eta,$$
$$\frac{z}{x} = -i\sin\varphi, \qquad \frac{s''}{s} = i\sin\eta,$$

unde, ubi ut ad expressiones reales perveniamus, loco α', α'', β, γ scribamus $i\alpha'$, $i\alpha''$, $-i\beta$, $-i\gamma$, substitutiones propositae in has abeunt:

$$\cos\varphi = \frac{\beta - \beta'\cos\eta - \beta''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$
$$\sin\varphi = \frac{\gamma - \gamma'\cos\eta - \gamma''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta}.$$

Porro aequationem:

$$\frac{ax^2 + bxy + cxz + dy^2 + eyz + fz^2}{xx} = \frac{GGss + G'G's's' + G''G''s''s''}{(\alpha s + \alpha' s' + \alpha'' s'')^2},$$

ubi rursus loco b, c, d, e, f scribamus ib, ic, $-d$, $-e$, $-f$, in hanc abire videmus:

$$a + b\cos\varphi + c\sin\varphi + d\cos\varphi^2 + e\cos\varphi\sin\varphi + f\sin\varphi^2 = \frac{GG - G'G'\cos\eta^2 - G''G''\sin\eta^2}{(\alpha - \alpha'\cos\eta - \alpha''\sin\eta)^2}.$$

Unde cum facile probetur, esse

$$\partial\varphi = \frac{\partial\gamma}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

sequitur transformatio illa:

$$\int \frac{\partial\varphi}{\sqrt{(a + b\cos\varphi + c\sin\varphi + d\cos\varphi^2 + e\cos\varphi\sin\varphi + f\sin\varphi^2)}}$$

$$= \int \frac{\partial\gamma}{\sqrt{(GG - G'G'\cos\eta^2 - G''G''\sin\eta^2)}} = \frac{1}{GG - G'G'} \int \frac{\partial\eta}{\sqrt{\left(1 - \frac{G''G'' - G'G'}{GG - G'G'}\sin\eta^2\right)}},$$

ubi integrale reductum formam assignatam habet. Hinc videmus, utriusque problematis solutiones alteram ex altera obtineri, ubi loco

$$\alpha',\ \alpha'',\ \beta,\ \gamma,\ b,\ c,\ d,\ e,\ f$$

scribatur respective:

$$i\alpha',\ i\alpha'',\ -i\beta,\ -i\gamma,\ ib,\ ic,\ -d,\ -e,\ -f,$$

posito $i = \sqrt{-1}$.

5.

De natura substitutionis

$$\cos\varphi = \frac{\beta - \beta'\cos\eta - \beta''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

$$\sin\varphi = \frac{\gamma - \gamma'\cos\eta - \gamma''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta},$$

et reductione integralis, cui inservit, fusius egi, cum per eandem substitutionem, sed binis simul variabilibus applicatam, etiam reductio proposita integralis duplicis succedat. Videbimus enim sequentibus, propositum integrale duplex

$$\int \frac{\partial\varphi\,\partial\psi}{A + B\cos\varphi + C\sin\varphi + (A' + B'\cos\varphi + C'\sin\varphi)\cos\psi + (A'' + B''\cos\varphi + C''\sin\varphi)\sin\psi}$$

per substitutiones

$$\cos\varphi = \frac{\beta - \beta'\cos\eta - \beta''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta} \quad\Bigg|\quad \cos\psi = \frac{a' - b'\cos\vartheta - c'\sin\vartheta}{a - b\cos\vartheta - c\sin\vartheta}$$

$$\sin\varphi = \frac{\gamma - \gamma'\cos\eta - \gamma''\sin\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta} \quad\Bigg|\quad \sin\psi = \frac{a'' - b''\cos\vartheta - c''\sin\vartheta}{a - b\cos\vartheta - c\sin\vartheta}$$

simul adhibitas transformari posse in formam simpliciorem

$$\int \frac{\partial\eta\,\partial\vartheta}{G - G'\cos\eta\cos\vartheta - G''\sin\eta\sin\vartheta}.$$

Quin adeo videbimus, ipsam hanc integralis duplicis transformationem ad eiusmodi binorum integralium simplicium reductionem, qualem supra exhibuimus, revocari.

Et haec de transformando duplici integrali quaestio, uti transformatio illa integralis simplicis, cum problemate algebraico convenit, ita ut ex alterius solutione alterius solutionem levissimis mutationibus factis petere liceat. Quod problema algebraicum hoc est, per substitutiones

$$\begin{array}{l|l} x = \alpha s + \alpha' s' + \alpha'' s'' & w = at + bu + cv \\ y = \beta s + \beta' s' + \beta'' s'' & w' = a't + b'u + c'v \\ z = \gamma s + \gamma' s' + \gamma'' s'' & w'' = a''t + b''u + c''v, \end{array}$$

quae identice efficiant

$$xx + yy + zz = ss + s's' + s's',$$
$$ww + w'w' + w''w'' = tt + uu + vv,$$

simul transformare expressionem

$$(Ax + By + Cz)w + (A'x + B'y + C'z)w' + (A''x + B''y + C''z)w''$$

in hanc simpliciorem:

$$Gst + G's'u + G''s''v.$$

Cuius problematis solutionem suscipiamus vel antequam ad transformationem integralis duplicis accedemus, atque monstremus, quomodo illa absoluta, confestim etiam hanc obtines: cum ut exemplo luculento transitum illum memorabilem ab altero ad alterum problema demonstremus, tum quia problema algebraicum elegentia quodammodo et symmetria calculi praevalet, et per se dignum est, in quod inquiratur.

6.

Expositis variis relationibus, quae in problemate algebraico inter octodecim coëfficientes substitutionum et tres quantitates G, G', G'' locum habent, invenientur primum harum quadrata GG, $G'G'$, $G''G''$ ut radices diversae aequationis cubicae:

$$\begin{aligned} & x^3 - x^2(AA + BB + CC + A'A' + B'B' + C'C' + A''A'' + B''B'' + C''C'') \\ & + x\left\{\begin{aligned} & (B'C'' - B''C')^2 + (B''C - BC'')^2 + (BC' - B'C)^2 \\ & + (C'A'' - C''A')^2 + (C''A - CA'')^2 + (CA' - C'A)^2 \\ & + (A'B'' - A''B')^2 + (A''B - AB'')^2 + (AB' - A'B)^2 \end{aligned}\right\} \\ & - \{A(B'C'' - B''C') + A'(B''C - BC'') + A''(BC' - B'C)\}^2 = 0. \end{aligned}$$

Quibus erutis, quadrata coëfficientium substitutionis nec non producta

$$\begin{array}{ccc|ccc} \alpha'\alpha'' & \alpha''\alpha & \alpha\alpha' & bc & ca & ab \\ \beta'\beta'' & \beta''\beta & \beta\beta' & b'c' & c'a' & a'b' \\ \gamma'\gamma'' & \gamma''\gamma & \gamma\gamma' & b''c'' & c''a'' & a''b'' \end{array}$$

per formulas rationales exhibentur. Utriusque autem substitutionis coëfficientes ope ipsarum G, G', G'' alterae per alteras idque variis modis linealiter exprimuntur.

Observabitur porro, expressiones

$$(Ax+By+Cz)^2+(A'x+B'y+C'z)^2+(A''x+By+C''z)^2,$$

$$Aw+A'w'+A''w'')^2+(Bw+B'w'+B''w'')^2+(Cw+C'w'+C''w'')^2$$

per easdem substitutiones, singulas singulis applicatas, transformari in has simpliciores:

$$GGss+G'G's's'+G''G''s''s'',$$

$$GGtt+G'G'uu+G''G''vv.$$

In utraque expressione reducta memoratu dignum est, coëfficientes GG, $G'G'$, $G''G''$ easdem esse, quod etiam inde patet, quod aequatio cubica, cuius illae radices sunt, immutata maneat, ubi constantium

$$\begin{matrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{matrix}$$

series horizontales et verticales inter se permutantur. Quod theorema geometricum suppeditat, ellipsoidas, quae ad coordinatas orthogonales relatae definiantur per aequationes

$$(Ax+By+Cz)^2+(A'x+B'y+C'z)^2+(A''x+B''y+C''z)^2=KK,$$

$$(Aw+A'w+A''w'')^2+(Bw+B'w'+B''w'')^2+(Cw+C'w'+C''w'')^2=KK,$$

easdem esse nec situ in spatio diversas, quippe utriusque inveniantur semiaxes principales $\frac{K}{G}$, $\frac{K}{G'}$, $\frac{K}{G''}$. Qua observatione problema propositum algebraicum revocatur ad indagationem axium principalium ellipsoidarum, quae aequationibus assignatis continentur.

Per easdem substitutiones invenitur, etiam expressionem

$$\begin{aligned} &[(B'C''-B''C')x+(C'A''-C''A')y+(A'B''-A''B')z]w \\ +&[(B''C-BC'')x+(C''A-CA'')y+(A''B-AB'')z]w' \\ +&[(BC'-B'C)x+(CA'-C'A)y+(AB'-A'B)z]w'' \end{aligned}$$

abire in hanc simpliciorem

$$G'G''st+G''Gs'u+GG's''v;$$

nec non ellipsoidas, quae ad coordinatas orthogonales relatae definiantur per aequationes:

$$\begin{aligned} &[(B'C''-B''C')x+(C'A''-C''A')y+(A'B''-A''B')z]^2 \\ +&[(B''C-BC'')x+(C''A-CA'')y+(A''B-AB'')z]^2 \\ +&[(BC'-B'C)x+(CA'-C'A)y+(AB'-A'B)z]^2=KK, \end{aligned}$$

$$\begin{aligned} &[(B'C''-B''C')w+(B''C-BC'')w+(BC'-B'C)w'']^2 \\ +&[(C'A''-C''A')w+(C''A-CA'')w+(CA'-C'A)w'']^2 \\ +&[(A'B''-A''B')w+(A''B-AB'')w+(AB'-A'B)w'']^2=KK \end{aligned}$$

per easdem substitutiones ad axes earum principales revocari, quae cum pro utraque inveniantur

$$\frac{K}{G'G''},\ \frac{K}{G''G},\ \frac{K}{GG'},$$

et haec ellipsoidae eaedem erunt nec nisi situ diversae.

7.

Absoluto problemate algebraico, ut inde transformationem integralis duplicis propositam eruamus, ponamus rursus

$$xx+yy+zz = ss+s's'+s''s'' = 0$$

nec non

$$ww+w'w'+w''w'' = tt+uu+vv = 0,$$

atque, ut supra, statuamus:

$$\begin{array}{l|l}
\frac{y}{x} = -i\cos\varphi & \frac{w'}{w} = -i\cos\psi \\
\frac{z}{x} = -i\sin\varphi & \frac{w''}{w} = -i\sin\eta \\
\frac{s'}{s} = \quad i\cos\eta & \frac{u}{t} = \quad i\cos\vartheta \\
\frac{s''}{s} = \quad i\sin\eta & \frac{v}{t} = \quad i\sin\vartheta
\end{array}$$

Unde, ubi rursus loco

$$\left.\begin{matrix} \alpha', & \alpha'', & \beta, & \gamma \\ b, & c, & a', & a'' \end{matrix}\right\}\text{ scribatur }\left\{\begin{matrix} i\alpha', & i\alpha'', & -i\beta, & -i\gamma \\ ib, & ic, & -i\alpha, & -i\alpha''. \end{matrix}\right.$$

e substitutionibus §. 7. adhibitis prodeunt:

$$\begin{array}{l|l}
\cos\varphi = \frac{\beta-\beta'\cos\eta-\beta''\sin\eta}{\alpha-\alpha'\cos\eta-\alpha''\sin\eta} & \cos\psi = \frac{a'-b'\cos\vartheta-c'\sin\vartheta}{a-b\cos\vartheta-c\sin\vartheta} \\
\sin\varphi = \frac{\gamma-\gamma'\cos\eta-\gamma''\sin\eta}{\alpha-\alpha'\cos\eta-\alpha''\sin\eta} & \sin\psi = \frac{a''-b''\cos\vartheta-c''\sin\vartheta}{a-b\cos\vartheta-c\sin\vartheta}.
\end{array}$$

Porro aequatio

$$\frac{(Ax+By+Cz)w+(A'x+B'y+C'z)w'+(A''x+B''y+C''z)w''}{wx}$$
$$= \frac{Gst+G's'u+G''s''v}{(\alpha s+\alpha's'+\alpha''s'')(at+bu+cv)},$$

ubi rursus loco

$$A,\ B,\ C,\ A',\ B',\ C',\ A'',\ B'',\ C''$$

scribatur:

$$A,\ iB,\ iC,\ iA',\ -B',\ -C',\ iA'',\ -B'',\ -C'',$$

in hanc abit:

$$A+B\cos\varphi+C\sin\psi+(A'+B'\cos\varphi+C'\sin\varphi)\cos\psi+(A''+B''\cos\varphi+C''\sin\varphi)\sin\psi$$
$$= \frac{G-G'\cos\eta\cos\vartheta-G''\sin\eta\sin\vartheta}{(\alpha-\alpha'\cos\eta-\alpha''\sin\eta)(a-b\cos\vartheta-c\sin\vartheta)}.$$

Unde, cum sit:

$$\partial\varphi = \frac{\partial\eta}{\alpha - \alpha'\cos\eta - \alpha''\sin\eta}$$

$$\partial\psi = \frac{\partial\vartheta}{a - b\cos\vartheta - c\sin\vartheta},$$

prodit transformatio quaesita integralis duplicis propositi:

$$\int\frac{\partial\varphi\,\partial\psi}{(A+B\cos\varphi+C\sin\varphi)+(A'+B'\cos\varphi+C'\sin\varphi)\cos\psi+(A''+B''\cos\varphi+C''\sin\varphi)\sin\psi}$$

$$=\int\frac{\partial\eta\,\partial\vartheta}{G-G'\cos\eta\cos\vartheta-G''\sin\eta\sin\vartheta}.$$

8.

Quemadmodum problema algebraicum ad aliud revocare licet, quod investigationem axium principalium ellipsoidarum concernit, ita etiam transformatio duplicis integralis, quae illi respondet, eo revocari potest, ut integralia simplicia

$$\int\frac{\partial\varphi}{\sqrt{[(A+B\cos\varphi+C\sin\varphi)^2-(A'+B'\cos\varphi+C'\sin\varphi)^2-(A''+B''\cos\varphi+C''\sin\varphi)^2]}},$$

$$\int\frac{\partial\psi}{\sqrt{[(A+A'\cos\psi+A''\sin\psi)^2-(B+B'\cos\psi+B''\sin\psi)^2-(C+C'\cos\psi+C''\sin\psi)^2]}},$$

per substitutiones assignatas transformentur in haec:

$$\int\frac{\partial\eta}{\sqrt{(GG-G'G'\cos\eta^2-G''G''\sin\eta^2)}},$$

$$\int\frac{\partial\vartheta}{\sqrt{(GG-G'G'\cos\vartheta^2-G''G''\sin\vartheta^2)}},$$

quae videmus nonnisi argumento differre, quemadmodum in illo problemate ellipsoidae propositae nonnisi situ differebant.

Hinc solutionem problematis propositi semper realem fore sequitur, ubi expressio

$$A+B\cos\varphi+C\sin\varphi+(A'+B'\cos\varphi+C'\sin\varphi)\cos\psi+(A''+B''\cos\varphi+C''\sin\varphi)\sin\psi$$

pro nullo angulorum φ, ψ valore reali evanescat, qui casus prae ceteris applicationem invenit. Posito enim

$$\frac{A''+B''\cos\varphi+C''\sin\varphi}{A'+B'\cos\varphi+C'\sin\varphi} = \operatorname{tang}\varepsilon$$

$$\frac{C+C'\cos\psi+C''\sin\psi}{B+B'\cos\psi+B''\sin\psi} = \operatorname{tang}\zeta,$$

expressio illa ita repraesentari potest:

$$A+B\cos\varphi+C\sin\varphi+\sqrt{((A'+B'\cos\varphi+C'\sin\varphi)^2)+(A''+B''\cos\varphi+C''\sin\varphi)^2}\cos(\psi-\varepsilon)$$

sive etiam

$$A+A'\cos\psi+A''\sin\psi+\sqrt{((B+B'\cos\psi+B''\sin\psi)^2+(C+C'\cos\psi+C''\sin\psi)^2}\cos(\varphi-\zeta)$$

quae ne pro ullo valore reali ipsorum φ, ψ evanescant, expressiones
$(A+B\cos\varphi+C\sin\varphi)^2-(A'+B'\cos\varphi+C'\sin\varphi)^2-(A''+B''\cos\varphi+C''\sin\varphi)^2$,
$(A+B'\cos\psi+A''\sin\psi)^2-(B+B'\cos\psi+B''\sin\psi)^2-(C+C'\cos\psi+C''\sin\psi)^2$
semper positivo valore gaudeant, necesse est. Quo casu substitutiones assignatas reales fore probavimus.

Per easdem substitutiones, quibus aequatio
$A+B\cos\varphi+C\sin\varphi+(A'+B'\cos\varphi+C'\sin\varphi)\cos\psi+(A''+B''\cos\varphi+C''\sin\varphi)\sin\psi=0$
in hanc abit:

$$G-G'\cos\eta\cos\vartheta-G''\sin\eta\sin\vartheta=0,$$

videmus, etiam aequationem differentialem

$$\frac{\partial\varphi}{\sqrt{[(A'+B'\cos\varphi+C'\sin\varphi)^2+(A''+B''\cos\varphi+C''\sin\varphi)^2-(A+B\cos\varphi+C\sin\varphi)^2]}}+$$
$$\frac{\partial\psi}{\sqrt{[(B+B'\cos\psi+B''\sin\psi)^2+(C+C'\cos\psi+C''\sin\psi)^2-(A+A'\cos\psi+A''\sin\psi)^2]}}=0$$

in hanc transformari:

$$\frac{\partial\eta}{\sqrt{(G'G'\cos\eta^2+G''G''\sin\eta^2-GG)}}+\frac{\partial\vartheta}{\sqrt{(G'G'\cos\vartheta^2+G''G''\sin\vartheta^2-GG)}}=0.$$

Facile autem probatur, aequationes illas finitas aequationum differentialium integralia completa esse. Unde theorematum inventorum verificatio obtinetur.

Nec non observabitur, posito

$$\begin{aligned}
\mathrm{P}=\;&B'C''-B''C'-(C'A''-C''A')\cos\varphi-(A'B''-A''B')\sin\varphi\\
&-\cos\psi\,[B''C-BC''-(C''A-CA'')\cos\varphi-(A''B-AB'')\sin\varphi]\\
&-\sin\psi\,[BC'-B'C-(CA'-C'A)\cos\varphi-(AB'-A'B)\sin\varphi],\\
\Sigma=\;&[B'C''-B''C'-(C'A''-C''A')\cos\varphi-(A'B''-A''B')\sin\varphi]^2\\
&-[B''C-BC''-(C''A-CA'')\cos\varphi-(A''B-AB'')\sin\varphi]^2\\
&-[BC'-B'C-(CA'-C'A)\cos\varphi-(AB'-A'B)\sin\varphi]^2,\\
\mathrm{T}=\;&[B'C''-B''C'-(B''C-BC'')\cos\psi-(BC'-B'C)\sin\psi]^2\\
&-[C'A''-C''A'-(C''A-CA'')\cos\psi-(CA'-C'A)\sin\psi]^2\\
&-[A'B''-A''B'-(A''B-AB'')\cos\psi-(AB'-A'B)\sin\psi]^2,
\end{aligned}$$

per easdem substitutiones nostras obtineri:

$$\int\frac{\partial\varphi\,\partial\psi}{\mathrm{P}}=\int\frac{\partial\eta\,\partial\vartheta}{G'G''-G''G\cos\eta\cos\vartheta-GG'\sin\eta\sin\vartheta},$$
$$\int\frac{\partial\varphi}{\Sigma^{\frac{1}{2}}}=\int\frac{\partial\eta}{\sqrt{(G'^2G''^2-G''^2G^2\cos\eta^2-G^2G'^2\sin\eta^2)}},$$
$$\int\frac{\partial\psi}{\mathrm{T}^{\frac{1}{2}}}=\int\frac{\partial\vartheta}{\sqrt{(G'^2G''^2-G''^2G^2\cos\vartheta^2-G^2G'^2\sin\vartheta^2)}}.$$

Quae antecedentibus iunctae sex transformationes memorabiles suppeditant, ad quas per easdem substitutiones pervenimus.

9.

Problema de duplici integrali transformando propositum etiam absque functionibus trigonometricis exhiberi potuisset. Facile enim intelligitur, eius in locum substitui posse sequens

Problema.

„**Integrale duplex indefinitum**

$$\int\frac{\partial x\,\partial y}{A+Bx+Cx^2+(A'+B'x+C'x^2)y+(A''+B''x+C''x^2)y^2},$$

per substitutiones formae

$$x=\frac{m+nt}{1+pt},\quad y=\frac{m'+n'u}{1+p'u}$$

transformare in hoc

$$\int\frac{\partial t\,\partial u}{D+Et^2+Ftu+Gu^2+Ht^2u^2},$$

cuius denominator terminis dimensionum imparum caret."

Praeplacuit tamen forma trigonometrica, quae in aliis quibusdam quaestionibus, de quibus in posterum agam, obvenit. Quamquam forma illa algebraica eo quoque nomine se commendat, quod solutione semper reali gaudet.

Jam ad solutionem quaestionum propositarum accedamus, et primum de problemate algebraico agam, e cuius deinde solutione propositam petamus duplicis integralis transformationem.

Problema I.

„**Proponitur, per substitutiones lineares:**

$$\begin{array}{l|l} x=\alpha s+\alpha' s'+\alpha'' s'' & w=at+bu+cv \\ y=\beta s+\beta' s'+\beta'' s'' & w'=a't+b'u+c'v \\ z=\gamma s+\gamma' s'+\gamma'' s'' & w''=a''t+b''u+c''v, \end{array}$$

quae identice efficiant:

$$xx+yy+zz=ss+s's'+s''s''$$
$$ww+w'w'+w''w''=tt+uu+vv,$$

transformare expressionem

$$(Ax+By+Cz)w+(A'x+B'y+C'z)w'+(A''x+B''y+C''z)w'')$$

in hanc simpliciorem

$$Gst + G's'u + G''s''v."$$

Solutio.

10.

E theoria transformationis systematis axium coordinatarum orthogonalium in aliud ejusmodi systema notae sunt relationes, quae inter coëfficientes substitutionum

$$1)\quad \begin{array}{l|l} x = \alpha s + \alpha' s' + \alpha'' s'' & w = at + bu + cv \\ y = \beta s + \beta' s' + \beta'' s'' & w' = a't + b'u + c'v \\ z = \gamma s + \gamma' s' + \gamma'' s'' & w'' = a''t + b''u + c''v \end{array}$$

locum habere debent, ut identice sit:

$$2)\quad \begin{array}{l} xx + yy + zz = ss + s's' + s''s'' \\ ww + w'w' + w''w'' = tt + uu + vv, \end{array}$$

sive

$$3)\quad \begin{array}{l} ss + s's' + s''s'' = (\alpha s + \alpha' s' + \alpha'' s'')^2 + (\beta s + \beta' s' + \beta'' s'')^2 + (\gamma s + \gamma' s' + \gamma'' s'')^2 \\ tt + uu + vv = (at + bu + cv)^2 + (a't + b'u + c'v)^2 + (a''t + b''u + c''v)^2; \end{array}$$

id quod aequationes conditionales poscit:

$$4)\quad \begin{array}{l|l} \alpha\alpha + \beta\beta + \gamma\gamma = 1 & aa + a'a' + a''a'' = 1 \\ \alpha'\alpha' + \beta'\beta' + \gamma'\gamma' = 1 & bb + b'b' + b''b'' = 1 \\ \alpha''\alpha'' + \beta''\beta'' + \gamma''\gamma'' = 1 & cc + c'c' + c''c'' = 1 \\ \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' = 0 & bc + b'c' + b''c'' = 0 \\ \alpha''\alpha + \beta''\beta + \gamma''\gamma = 0 & ca + c'a' + c''a'' = 0 \\ \alpha\alpha' + \beta\beta' + \gamma\gamma' = 0 & ab + a'b' + a''b'' = 0. \end{array}$$

Quarum relationum ope facile probantur aequationes:

$$5)\quad \begin{array}{l|l} s = \alpha x + \beta y + \gamma z & t = aw + a'w' + a''w'' \\ s' = \alpha' x + \beta' y + \gamma' z & u = bw + b'w' + b''w'' \\ s'' = \alpha'' x + \beta'' y + \gamma'' z & v = cw + c'w' + c''w'', \end{array}$$

quippe quae substitutis valoribus ipsarum x, y, z et w, w', w'' e 1) petitis, propter 4) identicae fiunt. Hinc cum e 2) fiat identice:

$$6)\quad \begin{array}{l} xx + yy + zz = (\alpha x + \beta y + \gamma z)^2 + (\alpha' x + \beta' y + \gamma' z)^2 + (\alpha'' x + \beta'' y + \gamma'' z)^2 \\ ww + w'w' + w''w'' = (aw + a'w' + a''w'')^2 + (bw + b'w' + b''w'')^2 + (cw + c'w' + c''w'')^2, \end{array}$$

sequuntur etiam:

$$7)\quad \begin{array}{l|l} \alpha\alpha + \alpha'\alpha' + \alpha''\alpha'' = 1 & aa + bb + cc = 1 \\ \beta\beta + \beta'\beta' + \beta''\beta'' = 1 & a'a' + b'b' + c'c' = 1 \\ \gamma\gamma + \gamma'\gamma' + \gamma''\gamma'' = 1 & a''a'' + b''b'' + c''c'' = 1 \end{array}$$

$$\begin{array}{l|l} \beta\gamma+\beta'\gamma'+\beta''\gamma''=0 & a'a''+b'b''+c'c''=0 \\ \gamma\alpha+\gamma'\alpha'+\gamma''\alpha''=0 & a''a+b''b+c''c=0 \\ \alpha\beta+\alpha'\beta'+\alpha''\beta''=0 & a\,a'+b\,b'+c\,c'=0. \end{array}$$

Expressiones ipsarum s, s', s'' per x, y, z atque ipsarum t, u, v per w, w', w'', quas formulae 5) suppeditant, etiam e 1) per methodum vulgarem resolutionis aequationum linearium petere licet. Expressionibus, quae inde sequuntur, cum illis comparatis, posito:

$$8)\quad \begin{aligned} \varepsilon &= \alpha(\beta'\gamma''-\beta''\gamma')+\beta(\gamma'\alpha''-\gamma''\alpha')+\gamma(\alpha'\beta''-\alpha''\beta') \\ e &= a(b'c''-b''c')+a'(b''c-bc'')+a''(bc'-b'c), \end{aligned}$$

obtinemus:

$$9)\quad \begin{array}{l|l} \varepsilon\alpha = \beta'\gamma''-\beta''\gamma' & ea = b'c''-b''c' \\ \varepsilon\beta = \gamma'\alpha''-\gamma''\alpha' & ea' = b''c-bc'' \\ \varepsilon\gamma = \alpha'\beta''-\alpha''\beta' & ea'' = bc'-b'c \\ \varepsilon\alpha' = \beta''\gamma-\beta\gamma'' & eb = c'a''-c''a' \\ \varepsilon\beta' = \gamma''\alpha-\gamma\alpha'' & eb' = c''a-ca'' \\ \varepsilon\gamma' = \alpha''\beta-\alpha\beta'' & eb'' = ca'-c'a \\ \varepsilon\alpha'' = \beta\gamma'-\beta'\gamma & ec = a'b''-a''b' \\ \varepsilon\beta'' = \gamma\alpha'-\gamma'\alpha & ec' = a''b-ab'' \\ \varepsilon\gamma'' = \alpha\beta'-\alpha'\beta & ec'' = ab'-a'b \end{array}$$

Ipsas ε, e invenimus e formulis identicis:

$$\begin{aligned} (\gamma''\alpha-\gamma\alpha'')(\alpha\beta'-\alpha'\beta)-(\gamma\alpha'-\gamma'\alpha)(\alpha''\beta-\alpha\beta'') &= \alpha\varepsilon \\ (c''a-ca'')(ab'-a'b)-(ca'-c'a)(a''b-ab'') &= ae, \end{aligned}$$

quae e 9) in has abeunt

$$\begin{aligned} \varepsilon\varepsilon(\beta'\gamma''-\beta''\gamma') &= \varepsilon^3\alpha = \varepsilon\alpha \\ ee(b'c''-b''c') &= e^3 a = ea, \end{aligned}$$

sive $\varepsilon\varepsilon = ee = 1$, unde cum signum ipsarum ε, e pro arbitrio assumere liceat, statuemus:

$$10)\quad \varepsilon = 1;\quad e = 1.$$

Quae abunde nota, ne quid desit, hic apposuimus. Ante omnia autem tenendum est, quo in sequentibus saepe utemur,

theorema,

„naturam coëfficientium substitutionum propositarum eam esse, ut datis aequationibus linearibus:

$$\begin{array}{l|l} X = \alpha G+\alpha' G'+\alpha'' G'' & W = a\,T+b\,U+c\,V \\ Y = \beta G+\beta' G'+\beta'' G'' & W' = a'\,T+b'\,U+c'\,V \\ Z = \gamma G+\gamma' G'+\gamma'' G'' & W'' = a'\,T+b''U+c''V, \end{array}$$

inde sequatur:

$$\begin{array}{l|l} G \;\;= \alpha\, X + \beta\, Y + \gamma\, Z & T = aW + a'W' + a''W'' \\ G' \;= \alpha' X + \beta' Y + \gamma' Z & U = bW + b'W' + b''W'' \\ G'' = \alpha'' X + \beta'' Y + \gamma'' Z & V = cW + c'W' + c''W'', \end{array}$$

et vice versa; simulque sit:

$$\begin{array}{l} XX \;+ YY \;\;+ ZZ \;\;\;= GG + G'G' + G''G'' \\ WW + W'W' + W''W'' = TT \;+ UU \;+ VV." \end{array}$$

11.

E substitutionibus 1) cum prodire debeat, quod est problema propositum:

11) $(Ax + By + Cz)w + (A'x + B'y + C'z)w' + (A''x + B''y + C''z)w''$
$= Gst + G's'u + G''s''u,$

locum habere debent aequationes conditionales:

$$12)\quad \begin{array}{l} A \;= G\alpha a \;+ G'\alpha' b + G''\alpha'' c \\ B \;= G\beta a \;+ G'\beta' b + G''\beta'' c \\ C \;= G\gamma a \;+ G'\gamma' b + G''\gamma'' c \\[1ex] A' = G\alpha a' + G'\alpha' b' + G''\alpha'' c' \\ B' = G\beta a' + G'\beta' b' + G''\beta'' c' \\ C' = G\gamma a' + G'\gamma' b' + G''\gamma'' c' \\[1ex] A'' = G\alpha a'' + G'\alpha' b'' + G''\alpha'' c'' \\ B'' = G\beta a'' + G'\beta' b'' + G''\beta'' c'' \\ C'' = G\gamma a'' + G'\gamma' b'' + G''\gamma'' c''. \end{array}$$

Quae novem aequationes, junctae duodecim 4) unam et viginti efficiunt aequationes conditionales, quibus octodecim coëfficientes substitutionum propositarum et tres quantitates G, G', G'' satisfacere debent. Quod problema est determinatum. Jam unius et viginti incognitarum aggrediamur determinationem, atque varias, quae inter eas locum habent, relationes exponamus.

12.

Per theorema §. 10. e formulis 12) prodeunt sequentes:

$$13)\quad \begin{array}{l|l} Ga \;\;= \alpha A \;+ \beta B \;+ \gamma C & G\alpha \;= aA + a'A' + a''A'' \\ Ga' \;= \alpha A' + \beta B' + \gamma C' & G\beta \;= aB + a'B' + a''B'' \\ Ga'' = \alpha A'' + \beta B'' + \gamma C'' & G\gamma \;= aC + a'C' + a''C'' \\[1ex] G'b \;\;= \alpha' A \;+ \beta' B \;+ \gamma' C & G'\alpha' = bA + b'A' + b''A'' \\ G'b' \;= \alpha' A' + \beta' B' + \gamma' C' & G'\beta' = bB + b'B' + b''B'' \\ G'b'' = \alpha' A'' + \beta' B'' + \gamma' C'' & G'\gamma' = bC + b'C' + b''C'' \end{array}$$

$$G''c = \alpha''A + \beta''B + \gamma''C \qquad G''\alpha'' = cA + c'A' + c''A''$$
$$G''c' = \alpha'B' + \beta''B' + \gamma''C' \qquad G''\beta'' = cB + c'B' + c''B''$$
$$G''c'' = \alpha''B'' + \beta''B'' + \gamma''C'' \qquad G''\gamma'' = cC + c'C' + c''C''.$$

Quibus formulis utriusque substitutionis coëfficientes ope quantitatum G, G', G'' alterae per alteras lineariter exprimuntur.

Alteram ejusmodi determinationem ex ipsis 13) per resolutionem aequationum linearium petere licet; e. g. e formulis, quibus a, a', a'' per α, β, γ exprimuntur, vice versa etiam α, β, γ per a, a', a'' determinantur. Qua ratione, posito brevitatis causa

14) $\triangle = A(B'C'' - B''C') + B(C'A'' - C''A') + C(A'B'' - A''B')$,

obtines e 13) sequens formularum systema:

15)
$$\frac{\triangle\alpha}{G} = (B'C'' - B''C')a + (B''C - BC'')a' + (BC' - B'C)a''$$
$$\frac{\triangle\beta}{G} = (C'A'' - C''A')a + (C''A - CA'')a' + (CA' - C'A)a''$$
$$\frac{\triangle\gamma}{G} = (A'B'' - A''B')a + (A''B - AB'')a' + (AB' - A'B)a''$$
$$\frac{\triangle\alpha'}{G'} = (B'C'' - B''C')b + (B''C - BC'')b' + (BC' - B'C)b''$$
$$\frac{\triangle\beta'}{G'} = (C'A'' - C''A')b + (C''A - CA'')b + (CA' - C'A)b''$$
$$\frac{\triangle\gamma'}{G'} = (A'B'' - A''B')b + (A''B - AB'')b' + (AB' - A'B)b''$$
$$\frac{\triangle\alpha''}{G''} = (B'C'' - B''C')c + (B''C - BC'')c' + (BC' - B'C)c''$$
$$\frac{\triangle\beta''}{G''} = (C'A'' - C''A')c + (C''A - CA'')c' + (CA' - C'A)c''$$
$$\frac{\triangle\gamma''}{G''} = (A'B'' - A''B')c + (A''B - AB'')c' + (AB' - A'B)c''$$
$$\frac{\triangle a}{G} = \alpha(B'C'' - B''C') + \beta(C'A'' - C''A') + \gamma(A'B'' - A''B')$$
$$\frac{\triangle a'}{G} = \alpha(B''C - BC'') + \beta(C''A - CA'') + \gamma(A''B - AB'')$$
$$\frac{\triangle a''}{G} = \alpha(BC' - B'C) + \beta(CA' - C'A) + \gamma(AB' - A'B)$$
$$\frac{\triangle b}{G'} = \alpha'(B'C'' - B''C') + \beta'(C'A'' - C''A') + \gamma'(A'B'' - A''B')$$
$$\frac{\triangle b'}{G'} = \alpha'(B''C - BC'') + \beta'(C''A - CA'') + \gamma'(A''B - B''A)$$
$$\frac{\triangle b''}{G'} = \alpha'(BC' - B'C) + \beta'(CA' - C'A) + \gamma'(AB' - A'B)$$

$$\frac{\triangle c}{G''} = \alpha''(B'C''-B''C') + \beta''(C'A''-C''A') + \gamma''(A'B''-A''B')$$
$$\frac{\triangle c'}{G''} = \alpha''(B''C-BC'') + \beta''(C''A-CA'') + \gamma''(A''B-AB'')$$
$$\frac{\triangle c''}{G''} = \alpha''(BC'-B'C) + \beta''(CA'-C'A) + \gamma''(AB'-A'B).$$

E quibus formulis rursus per theorema §i 10. derivantur sequentes:

$$16)\quad \frac{B'C''-B''C'}{\triangle} = \frac{\alpha a}{G} + \frac{\alpha' b}{G'} + \frac{\alpha'' c}{G''}$$
$$\frac{C'A''-C''A'}{\triangle} = \frac{\beta a}{G} + \frac{\beta' b}{G'} + \frac{\beta'' c}{G''}$$
$$\frac{A'B''-A''B'}{\triangle} = \frac{\gamma a}{G} + \frac{\gamma' b}{G'} + \frac{\gamma'' c}{G''}$$
$$\frac{B''C-BC''}{\triangle} = \frac{\alpha a'}{G} + \frac{\alpha' b'}{G'} + \frac{\alpha'' c'}{G''}$$
$$\frac{C''A-CA''}{\triangle} = \frac{\beta a'}{G} + \frac{\beta' b'}{G'} + \frac{\beta'' c'}{G''}$$
$$\frac{A''B-AB''}{\triangle} = \frac{\gamma a'}{G} + \frac{\gamma' b'}{G'} + \frac{\gamma'' c'}{G''}$$
$$\frac{BC'-B'C}{\triangle} = \frac{\alpha a''}{G} + \frac{\alpha' b''}{G'} + \frac{\alpha'' c''}{G''}$$
$$\frac{CA'-C'A}{\triangle} = \frac{\beta a''}{G} + \frac{\beta' b''}{G'} + \frac{\beta'' c''}{G''}$$
$$\frac{AB'-A'B}{\triangle} = \frac{\gamma a''}{G} + \frac{\gamma' b''}{G'} + \frac{\gamma'' c''}{G''}.$$

Valores ipsarum $B'C''-B''C'$, $C'A''-C''A'$ etc. etiam directe e 12) derivare licet. Fit exempli gratia e formulis 12):

$$B' = G\beta a' + G'\beta' b' + G''\beta'' c'$$
$$C' = G\gamma a' + G'\gamma' b' + G''\gamma'' c'$$
$$B'' = G\beta a'' + G'\beta' b'' + G''\beta'' c''$$
$$C'' = G\gamma a'' + G'\gamma' b'' + G''\gamma'' c'',$$

prima et postrema, secunda et tertia in se ductis et subductione facta:

$$B'C''-B''C' = G'G''(\beta'\gamma''-\beta''\gamma')(b'c''-b''c')$$
$$+ G''G(\beta''\gamma-\beta\gamma'')(c'a''-c''a')$$
$$+ GG'(\beta\gamma'-\beta'\gamma)(a'b''-a''b'),$$

sive e 9):

$$B'C''-B''C' = G'G''\alpha a + G''G\alpha' b + GG'\alpha'' c;$$

qua comparata cum formula 16):

$$B'C''-B''C' = \frac{\triangle\alpha a}{G} + \frac{\triangle\alpha' b}{G'} + \frac{\triangle\alpha'' c}{G''},$$

prodit:

17) $\triangle = A(B'C''-B''C')+B(C'A''-C''A')+C(A'B''-A''B') = GG'G''$.

Iam accedamus ad alia formularum systemata.

13.

Ponatur brevitatis causa:

$$18)\quad \begin{array}{ll|ll} l &= AA+A'A'+A''A'' & p &= AA+BB+CC \\ m &= BB+B'B'+B''B'' & p' &= A'A'+B'B'+C'C' \\ n &= CC+C'C'+C''C'' & p'' &= A''A''+B''B''+C''C'' \\ l' &= BC+B'C'+B''C'' & q &= A'A''+B'B''+C'C'' \\ m' &= CA+C'A'+C''A'' & q' &= A''A+B''B+C''C \\ n' &= AB+A'B'+A''B'' & q'' &= AA'+BB'+CC'; \end{array}$$

unde etiam erit:

$$19)\quad \begin{aligned} mn-l'l' &= (B'C''-B''C')^2+(B''C-BC'')^2+(BC'-B'C)^2 \\ nl-m'm' &= (C'A''-C''A')^2+(C''A-CA'')^2+(CA'-C'A)^2 \\ lm-n'n' &= (A'B''-A''B')^2+(A''B-AB'')^2+(AB'-A'B)^2 \\ p'p''-qq &= (B'C''-B''C')^2+(C'A''-C''A')^2+(A'B''-A''B')^2 \\ p''p-q'q' &= (B''C-BC'')^2+(C''A-CA'')^2+(A''B-AB'')^2 \\ pp'-q''q'' &= (BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2; \end{aligned}$$

porro:

$$20)\quad \begin{array}{l|l} m'n'-ll' = & q'q''-pq = \\ (C'A''-C''A')(A'B''-A''B')+ & (B''C-BC'')(BC'-B'C)+ \\ (C''A-CA'')(A''B-AB'')+ & (C''A-CA'')(CA'-C'A)+ \\ (CA'-C'A)(AB'-A'B) & (A''B-AB'')(AB'-A'B) \\ n'l'-mm' = & q''q-p'q' = \\ (A'B''-A''B')(B'C''-B''C')+ & (BC'-B'C)(B'C''-B''C')+ \\ (A''B-AB'')(B''C-BC'')+ & (CA'-C'A)(C'A''-C''A')+ \\ (AB'-A'B)(BC'-B'C) & (AB'-A'B)(A'B''-A''B') \\ l'm'-nn' = & qq'-p''q'' = \\ (B'C''-B''C')(C'A''-C''A')+ & (B'C''-B''C')(B''C-BC'')+ \\ (B''C-BC'')(C''A-CA'')+ & (C'A''-C''A')(C''A-CA'')+ \\ (BC'-B'C)(CA'-C'A) & (A'B''-A''B')(A''B-AB''); \end{array}$$

nec non:

$$21)\quad \begin{aligned} \triangle\triangle &= lmn-ll'l'-mm'm'-nn'n'+2l'm'n' \\ &= pp'p''-pqq-p'q'q'-p''q''q''+2qq'q''. \end{aligned}$$

Quae ommia rursus per ipsas G, G', G'' et coëfficientes substitutionum exprimamus, quod ex antecedentibus sine negotio fit.

Ac primum e formulis 12) per theorema §i. 10. prodit:

$$\begin{array}{rl|l} 22) & l=GG\alpha\alpha+G'G'\alpha'\alpha'+G''G''\alpha''\alpha'' & p\;=GGaa\;+G'G'bb\;+G''G''cc \\ & m=GG\beta\beta+G'G'\beta'\beta'+G''G''\beta''\beta'' & p'\;=GGa'a'\;+G'G'b'b'\;+G''G''c'c' \\ & n=GG\gamma\gamma+G'G'\gamma'\gamma'+G''G''\gamma''\gamma'' & p''=GGa''a''+G'G'b''b''+G''G''c''c'', \end{array}$$

ac simili modo e 15):

$$\begin{array}{rl|l} 23) & \frac{mn-l'l'}{\triangle\triangle}=\frac{\alpha\alpha}{GG}+\frac{\alpha'\alpha'}{G'G'}+\frac{\alpha''\alpha''}{G''G''} & \frac{p'p''-qq}{\triangle\triangle}=\frac{aa}{GG}+\frac{bb}{G'G'}+\frac{cc}{G''G''} \\ & \frac{nl-m'm'}{\triangle\triangle}=\frac{\beta\beta}{GG}+\frac{\beta'\beta'}{G'G'}+\frac{\beta''\beta''}{G''G''} & \frac{p''p-q'q'}{\triangle\triangle}=\frac{a'a'}{GG}+\frac{b'b'}{G'G'}+\frac{c'c'}{G''G''} \\ & \frac{lm-n'n'}{\triangle\triangle}=\frac{\gamma\gamma}{GG}+\frac{\gamma'\gamma'}{G'G'}+\frac{\gamma''\gamma''}{G''G''} & \frac{pp'-q''q''}{\triangle\triangle}=\frac{a''a''}{GG}+\frac{b''b''}{G'G'}+\frac{c''c''}{G''G''}. \end{array}$$

Porro e 12) facile derivantur sequentes:

$$\begin{array}{rl|l} 24) & l'=GG\beta\gamma+G'G'\beta'\gamma'+G''G''\beta''\gamma'' & q\;=GGa'a''+G'G'b'b''+G''G''c'c'' \\ & m'=GG\gamma\alpha+G'G'\gamma'\alpha'+G''G''\gamma''\alpha'' & q'\;=GGa''a+G'G'b''b+G''G''c''c \\ & n'=GG\alpha\beta+G'G'\alpha'\beta'+G''G''\alpha''\beta'' & q''=GGaa'+G'G'bb'+G''G''cc', \end{array}$$

ac simili modo e 15):

$$\begin{array}{rl|l} 25) & \frac{m'n'-ll'}{\triangle\triangle}=\frac{\beta\gamma}{GG}+\frac{\beta'\gamma'}{G'G'}+\frac{\beta''\gamma''}{G''G''} & \frac{q'q''-pq}{\triangle\triangle}=\frac{a'a''}{GG}+\frac{b'b''}{G'G'}+\frac{c'c''}{G''G''} \\ & \frac{n'l'-mm'}{\triangle\triangle}=\frac{\gamma\alpha}{GG}+\frac{\gamma'\alpha'}{G'G'}+\frac{\gamma''\alpha''}{G''G''} & \frac{q''q-p'q'}{\triangle\triangle}=\frac{a''a}{GG}+\frac{b''b}{G'G'}+\frac{c''c}{G''G''} \\ & \frac{l'm'-nn'}{\triangle\triangle}=\frac{\alpha\beta}{GG}+\frac{\alpha'\beta'}{G'G'}+\frac{\alpha''\beta''}{G''G''} & \frac{qq'-p''q''}{\triangle\triangle}=\frac{aa'}{GG}+\frac{bb'}{G'G'}+\frac{cc'}{G''G''}, \end{array}$$

Sequitur porro e 22):

$$26)\quad \begin{aligned} GG+G'G'+G''G'' &= ll+mm+nn \\ &= pp+p'p'+p''p'', \end{aligned}$$

eodemque modo e 23), cum sit $\triangle=GG'G''$:

$$27)\quad \begin{aligned} G'G'G''G''+G''G''GG+GGG'G' &= mn+nl+lm-l'l'-m'm'-n'n' \\ &= p'p''+p''p+pp'-qq-q'q'-q''q''. \end{aligned}$$

Adnotemus adhuc, e formulis 22), 24) recte dispositis per theorema §i 10. erui sequentes:

$$\begin{array}{rl|l} 28) & GG\alpha\;=l\alpha+n'\beta+m'\gamma & GGa\;=pa+q''a'+q'a'' \\ & GG\beta\;=n'\alpha+m\beta+l'\gamma & GGa'\;=q''a+p'a'+qa'' \\ & GG\gamma\;=m'\alpha+l'\beta+n\gamma & GGa''=q'a+qa'+p''a'' \\ & G'G'\alpha'=l\alpha'+n'\beta'+m'\gamma' & G'G'b=pb+q''b'+q'b'' \\ & G'G'\beta'=n'\alpha'+m\beta'+l'\gamma' & G'G'b'=q''b+p'b'+qb'' \\ & G'G'\gamma'=m'\alpha'+l'\beta'+n\gamma' & G'G'b''=q'b+qb'+p''b'' \end{array}$$

$$\begin{array}{l|l} G''G''\alpha'' = l\alpha'' + n'\beta'' + m'\gamma'' & G''G''c = pc + q''c' + q'c'' \\ G''G''\beta'' = n'\alpha'' + m\beta'' + l'\gamma'' & G''G''c' = q''c + p'c' + q'c'' \\ G''G''\gamma'' = m'\alpha'' + l'\beta'' + n\gamma'' & G''G''c'' = q'c + qc' + p''c''. \end{array}$$

Quarum exempli gratia prima, quarta, septima alterius systematis per dictum theorema ex his fluunt, quas e formulis 22), 24) eligimus:

$$\begin{aligned} l &= \alpha . GG\alpha + \alpha'.G'G'\alpha' + \alpha''.G''G''\alpha'' \\ n' &= \beta . GG\alpha + \beta'.G'G'\alpha' + \beta''.G''G''\alpha'' \\ m' &= \gamma . GG\alpha + \gamma'.G'G'\alpha' + \gamma''.G''G''\alpha'', \end{aligned}$$

similique modo reliquae 28) eruuntur.

E 28) rursus per idem theorema fit:

$$\begin{aligned} 29)\quad ll + n'n' + m'm' &= G^4\alpha\alpha + G'^4\alpha'\alpha' + G''^4\alpha''\alpha'' \\ mm + l'l' + n'n' &= G^4\beta\beta + G'^4\beta'\beta' + G''^4\beta''\beta'' \\ nn + m'm' + l'l' &= G^4\gamma\gamma + G'^4\gamma'\gamma' + G''^4\gamma''\gamma'' \\ pp + q''q'' + q'q' &= G^4 aa + G'^4 bb + G''^4 cc \\ p'p' + qq + q''q'' &= G^4 a'a' + G'^4 b'b' + G''^4 c'c' \\ p''p'' + q'q' + qq &= G^4 a''a'' + G'^4 b''b'' + G''^4 c''c''; \end{aligned}$$

Quibus aliae variae addi possunt. Similia formularum systemata e formulis 23), 25) derivare licet. Quae tamen ex antecedentibus etiam fluunt ope theorematis generalis sequentis. Comparatis enim inter se formulis 12) et 16), quarum alterutris, advocatis insuper 4), coëfficientes substitutionum et ipsas G, G', G'' determinare licet, sponte prodit

theorema,

„e qualibet formularum propositarum derivari posse alteram, si in locum quantitatum

$$\begin{array}{ccc} A, & B, & C, \\ A', & B', & C', \\ A'', & B'', & C'', \\ G, & G', & G'' \end{array}$$

substituantur respective sequentes:

$$\begin{array}{ccc} \dfrac{B'C''-B''C'}{\Delta}, & \dfrac{C'A''-C''A'}{\Delta}, & \dfrac{A'B''-A''B'}{\Delta} \\ \dfrac{B''C-BC''}{\Delta}, & \dfrac{C''A-CA''}{\Delta}, & \dfrac{A''B-AB''}{\Delta} \\ \dfrac{BC'-B'C}{\Delta}, & \dfrac{CA'-C'A}{\Delta}, & \dfrac{AB'-A'B}{\Delta} \\ \dfrac{1}{G}, & \dfrac{1}{G'}, & \dfrac{1}{G''}; \end{array}$$

unde e. g. etiam pro $\triangle$ ponendum $\frac{1}{\triangle}$. Quod patet reciprocum esse, id est, ubi illa in haec abeant, simul etiam haec in illa mutari."

E quo theoremate memorabili formulae inventae alteram statim ei respondentem adiungere licet. Quemadmodum formulae 22) et 23), 24) et 25) per theorema illud alterae ex alteris derivantur. Cui tamen negotio singulis casibus supersedemus.

Per substitutiones propositas e formulis 12) fit:

$$[Ax+By+Cz]w+[A'x+B'y+C'z]w'+[A''x+B''y+C''z]w'' = Gst+G's'u+G''s''v;$$

per easdem substitutiones e formulis 16) sive ex aequatione illa per dictum theorema altera sequitur aequatio ei respondens:

$$30)\quad \begin{aligned}&[(B'C''-B''C')x+(C'A''-C''A')y+(A'B''-A''B')z]w \;+\\ &[(B''C-BC'')\,x+(C''A-CA'')\,y+(A''B-AB'')\,z]w' \;+\\ &[(BC'-B'C)\,x+(CA'-C'A)\,y+(AB'-A'B)\,z]w''\\ &=\triangle\left[\frac{st}{G}+\frac{s'u}{G'}+\frac{s''v}{G''}\right]=G'G''st+G''Gs'u+GG's''v.\end{aligned}$$

Ita per easdem substitutiones binas simul effici videmus transformationes.

14.

Postquam antecedentibus relationes praecipuas et elegantiores, quae inter quantitates quaesitas et datas locum habent, collegimus, iam sine negotio idque variis modis ex iis ipsi incognitarum valores fluunt.

Formulis 26), 27), 17) quantitatum GG, $G'G'$, $G''G''$ summam, summam productorum e binis, ipsarumque productum exhibuimus, unde aequationem cubicam assignare possumus, cuius quantitates illae radices sint, eaeque radices diversae. Quae per formulas allegatas, advocata 21), fit:

$$31)\quad \begin{aligned}&x^3-x^2[l+m+n]+x[mn+nl+lm-l'l'-m'm'-n'n']\\ &-[lmn-ll'l'-mm'm'-nn'n'+2l'm'n']=0,\end{aligned}$$

sive etiam:

$$32)\quad \begin{aligned}&x^3-x^2[p+p'+p'']+x[p'p''+p''p+pp'-qq-q'q'-q''q'']\\ &-[pp'p''-pqq-p'q'q'-p''q''q''+2qq'q'']=0;\end{aligned}$$

quas aequationes etiam hunc in modum repraesentare licet:

$$33)\quad \begin{aligned}&(x-l)(x-m)(x-n)-l'l'(x-l)-m'm'(x-m)-n'n'(x-n)\\ &-2l'm'n'=0\end{aligned}$$

$$34)\quad \begin{aligned}&(x-p)(x-p')(x-p'')-qq(x-p)-q'q'(x-p')-q''q''(x-p'')\\ &-2qq'q''=0.\end{aligned}$$

Quae e formulis 18), 19), 21) in hanc abeunt:

$$35)\quad x^3 - x^2[AA+BB+CC+A'A'+B'B'+C'C'+A''A''+B''B''+C''C'']$$
$$+x\begin{cases}(B'C''-B''C')^2+(C'A''-C''A')^2+(A'B''-A''B')^2+\\(B''C-BC'')^2+(C''A-CA'')^2+(A''B-AB'')^2+\\(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2\end{cases}$$
$$-[A(B'C''-B''C')+B(C'A''-C''A')+C(A'B''-A''B')]^2=0.$$

A cuius aequationis cubicae resolutione totum maxime problema pendet; quippe cuius inventis radicibus GG, $G'G'$, $G''G''$, quadrata coëfficientium substitutionum propositarum rationaliter exprimuntur, unde ut ipsi earum eruantur valores, tantum radicis quadraticae extractione opus est.

Eligamus e. g., ut valorem ipsius $\alpha\alpha$ eruamus, e formulis 7), 22) 23) sequentes:

$$\alpha\alpha+\alpha'\alpha'+\alpha''\alpha''=1$$
$$GG\alpha\alpha+G'G'\alpha'\alpha'+G''G''\alpha''\alpha''=l$$
$$\frac{\alpha\alpha}{GG}+\frac{\alpha'\alpha'}{G'G'}+\frac{\alpha''\alpha''}{G''G''}=\frac{mn-l'l'}{\Delta\Delta},$$

quarum postrema etiam hunc in modum repraesentari potest:

$$G'G'G''G''\alpha\alpha+G''G''GG\alpha'\alpha'+GGG'G'\alpha''\alpha''=mn-l'l'.$$

Cui addatur prima ducta in $-GG(G'G'+G''G'')$, secunda ducta in GG; prodit:

$$[G^4-G^2(G'G'+G''G'')+G'G'G''G'']\alpha\alpha=G^2l-G^2(G'G'+G''G'')+mn-l'l',$$

unde cum sit

$$GG+G'G'+G''G''=l+m+n,$$

obtines:

$$\alpha\alpha=\frac{(GG-m)(GG-n)-l'l'}{(GG-G'G')(GG-G''G'')}.$$

In locum aequationis tertiae etiam haec substitui potest, e 29) petita:

$$G^4\alpha\alpha+G'^4\alpha'\alpha'+G''^4\alpha''\alpha''=ll+m'm'+n'n',$$

qua iuncta primae ductae in $G'G'G''G''$ et secundae ductae in $-(G'G'-G''G'')$, obtines

$$(GG-G'G')(GG-G''G'')\alpha\alpha=(G'G'-l)(G''G''-l)+m'm'+n'n',$$

sive

$$36)\quad \alpha\alpha=\frac{(l-G'G')(l-G''G'')+m'm'+n'n'}{(GG-G'G')(GG-G''G'')}.$$

Utrique autem ipsius $\alpha\alpha$ valores inventi e 26), 27) facile inter se conveniunt.

Hac ratione, cognitis ipsis GG, $G'G'$, $G''G''$, quadrata coëfficientium quaesitarum nanciscimur per formulas sequentes:

$$37)\quad \begin{aligned}
\alpha\alpha &= \frac{(GG-m)(GG-n)-l'l'}{(GG-G'G')(GG-G''G'')} & aa &= \frac{(GG-p')(GG-p'')-qq}{(GG-G'G')(GG-G''G'')}\\
\alpha'\alpha' &= \frac{(G'G'-m)(G'G'-n)-l'l'}{(G'G'-G''G'')(G'G'-GG)} & bb &= \frac{(G'G'-p')(G'G'-p'')\,qq}{(G'G'-G''G'')(G'G'-GG)}\\
\alpha''\alpha'' &= \frac{(G''G''-m)(G''G''-n)-l'l'}{(G''G''-GG)(G''G''-G'G')} & cc &= \frac{(G''G''-p')(G''G''-p'')-qq}{(G''G''-GG)(G''G''-G'G')}\\
\beta\beta &= \frac{(GG-n)(GG-l)-m'm'}{(GG-G'G')(GG-G''G'')} & a'a' &= \frac{(GG-p'')(GG-p)-q'q'}{(GG-G'G')(GG-G''G')}\\
\beta'\beta' &= \frac{(G'G'-n)(G'G'-l)-m'm'}{(G'G'-G''G'')(G'G'-GG)} & b'b' &= \frac{(G'G'-p'')(G'G'-p)-q'q'}{(G'G-G''G'')(G'G'-GG)}\\
\beta''\beta'' &= \frac{(G''G''-n)(G''G''-l)-m'm'}{(G''G''-GG)(G''G''-G'G')} & c'c' &= \frac{(G''G''-p'')(G''G''-p)-q'q'}{(G''G''-GG)(G''G''-G'G')}\\
\gamma\gamma &= \frac{(GG-l)(GG-m)-n'n'}{(GG-G'G')(GG-G''G'')} & a''a'' &= \frac{(GG-p)(GG-p')-q''q''}{(GG-G'G')(GG-G''G'')}\\
\gamma'\gamma' &= \frac{(G'G'-l)(G'G'-m)-n'n'}{(G'G'-G''G'')(G'G'-GG)} & b''b'' &= \frac{(G'G'-p)(G'G'-p')-q''q''}{(G'G'-G''G'')(G'G'-GG)}\\
\gamma''\gamma'' &= \frac{(G''G''-l)(G''G''-m)-n'n'}{(G''G''-GG)(G''G''-G'G')} & c''c'' &= \frac{(G''G''-p)(G''G''-p')-q''q''}{(G''G''-GG)(G''G''-G'G')}.
\end{aligned}$$

Uti quantitatem $\alpha\alpha$ sub alia forma 36) exhibuimus, ita etiam reliquis formam similem assignare licet, cui, cum in promtu sit, supersedemus.

His inventis, ipsas coëfficientes quaesitas per extractionem radicis quadraticae eruimus; signa autem radicum non omnia pro arbitrio assumere licet. Videbimus enim, non modo $\alpha\alpha$, sed etiam producta $\alpha\beta$, $\alpha\gamma$, ope ipsarum GG, $G'G'$, $G''G''$, rationaliter exprimi posse, unde patet, signo unius e quantitatibus α, β, γ pro arbitrio assumto, reliquarum signa determinata esse.

Producta illa $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$ eorumque similia ex antecedentibus facile invenimus. Exempli gratia, ut eruatur productum $\beta\gamma$, eligimus e 7), 24), 25) sequentes formulas:

$$\begin{aligned}
\beta\gamma + \beta'\gamma' + \beta''\gamma'' &= 0\\
GG\beta\gamma + G'G'\beta'\gamma' + G''G''\beta''\gamma'' &= l'\\
\frac{\beta\gamma}{GG} + \frac{\beta'\gamma'}{G'G'} + \frac{\beta''\gamma''}{G''G''} &= \frac{m'n'-ll'}{\triangle},
\end{aligned}$$

quarum postremam, cum sit $\triangle = GG'G''$, rursus ita exhibemus:

$$G'G'G''G''\beta\gamma + G''G''GG\beta'\gamma' + GGG'G'\beta''\gamma'' = m'n'-ll'.$$

Qua addita primae ductae in $-GG(G'G'+G''G'')$ et secundae ductae in GG, prodit:

$$(GG-G'G')(GG-G''G'')\beta\gamma = l'(GG-l)+m'n',$$

sive
$$\beta\gamma = \frac{l'(GG-l)+m'n'}{(GG-G'G')(GG-G''G'')}.$$

Hac ratione sequens nanciscimur formularum systema:

38)
$$\beta\gamma = \frac{l'(GG-l)+m'n'}{(GG-G'G')(GG-G''G'')} \qquad a'a'' = \frac{q(GG-p)+q'q''}{(GG-G'G')(GG-G''G'')}$$
$$\beta'\gamma' = \frac{l'(G'G'-l)+m'n'}{(G'G'-G''G'')(G'G'-GG)} \qquad b'b'' = \frac{q(G'G'-p)+q'q''}{(G'G'-G''G'')(G'G'-GG)}$$
$$\beta''\gamma'' = \frac{l'(G''G''-l)+m'n'}{(G''G''-GG)(G''G''-G'G')} \qquad c'c'' = \frac{q(G''G''-p)+q'q''}{(G''G''-GG)(G''G''-G'G')}$$
$$\gamma\alpha = \frac{m'(GG-m)+n'l'}{(GG-G'G')(GG-G''G'')} \qquad a''a = \frac{q'(GG-p')+q''q}{(GG-G'G')(GG-G''G'')}$$
$$\gamma'\alpha' = \frac{m'(G'G'-m)+n'l'}{(G'G'-G''G'')(G'G'-GG)} \qquad b''b = \frac{q'(G'G'-p')+q''q}{(G'G'-G''G'')(G'G'-GG)}$$
$$\gamma''\alpha'' = \frac{m'(G''G''-m)+n'l'}{(G''G''-GG)(G''G''-G'G')} \qquad c''c = \frac{q'(G''G''-p')+q''q}{(G''G''-GG)(G''G''-G'G')}$$
$$\alpha\beta = \frac{n'(GG-n)+l'm'}{(GG-G'G')(GG-G''G'')} \qquad aa' = \frac{q''(GG-p'')+qq'}{(GG-G'G')(GG-G''G'')}$$
$$\alpha'\beta' = \frac{n'(G'G'-n)+l'm'}{(G'G'-G''G'')(G'G'-GG)} \qquad bb' = \frac{q''(G'G'-p'')+qq'}{(G'G'-G''G'')(G'G'-GG)}$$
$$\alpha''\beta'' = \frac{n'(G''G''-n)+l'm'}{(G''G''-GG)(G''G''-G'G')} \qquad cc' = \frac{q''(G''G''-p'')+qq'}{(G''G''-GG)(G''G''-G'G')}.$$

Ex his formulis videmus, determinatis signis ipsarum α, α', α'' et a, b, c, reliquarum etiam signa determinata esse. Neque illa omnino pro arbitrio assumere licet, ubi, ut supra fecimus 10), statuere placet $\varepsilon = 1$, $e = 1$; quippe quo facto etiam e quantitatibus α, α', α'' nec non e quantitatibus a, b, c unius signum per signa duarum reliquarum determinatur.

Adnotemus adhuc, quo methodorum, quibus uti licet, varietas demonstretur, omnia, quae ad resolutionem problematis necessaria sint, etiam e formulis 28) peti potuisse. Eligamus e. g. aequationes tres primas alterius systematis, quas ita exhibemus:
$$\begin{aligned} 0 &= (l-GG)\alpha + n'\beta + m'\gamma \\ 0 &= n'\alpha + (m-GG)\beta + l'\gamma \\ 0 &= m'\alpha + l'\beta + (n-GG)\gamma, \end{aligned}$$
e quibus, eliminatis α, β, γ per regulas notas, primum obtinemus:
$$(l-GG)(m-GG)(n-GG) - l'l'(l-GG) - m'm'(m-GG) - n'n'(n-GG) + 2l'm'n' = 0,$$
quae aequatio cubica, e cuius resolutione GG prodit, eadem est atque illa supra inventa 33). Eadem methodo e reliquis formulis 28) inveniuntur $G'G'$, $G''G''$ eiusdem aequationis cubicae radices esse.

Porro ex aequatione secunda et tertia sequuntur proportiones:

$$\alpha:\beta:\gamma = \alpha\alpha:\alpha\beta:\alpha\gamma =$$
$$(m-GG)(n-GG)-l'l':l'm'-n'(n-GG):n'l'-m'(m-GG);$$

e tertia et prima:

$$\beta:\gamma:\alpha = \beta\beta:\beta\gamma:\beta\alpha =$$
$$(n-GG)(l-GG)-m'm':m'n'-l'(l-GG):l'm'-n'(n-GG);$$

e prima et secunda:

$$\gamma:\alpha:\beta = \gamma\gamma:\gamma\alpha:\gamma\beta =$$
$$(l-GG)(m-GG)-n'n':n'l'-m'(m-GG):m'n'-l'(l-GG).$$

Unde etiam:

$$\alpha\alpha:\beta\beta:\gamma\gamma =$$
$$(m-GG)(n-GG)-l'l':(n-GG)(l-GG)-m'm':(l-GG)(m-GG)-n'n'.$$

Jam vero est

$$(m-GG)(n-GG)+(n-GG)(l-GG)+(l-GG)(m-GG)-l'l'-m'm'-n'n'$$

aequale differentiali expressionis:

$$(x-l)(x-m)(x-n)-l'(x-l)-m'(x-m)-n'(x-n)-2l'm'n'$$
$$=(x-GG)(x-G'G')(x-G''G''),$$

secundum x sumto, siquidem post differentiationem $x = GG$ ponitur, ideoque etiam aequale expressioni

$$(GG-G'G')(GG-G''G'').$$

Unde cum sit

$$\alpha\alpha+\beta\beta+\gamma\gamma = 1,$$

eruimus:

$$\alpha\alpha = \frac{(GG-m)(GG-n)-l'l'}{(GG-G'G')(GG-G''G'')},$$

quod cum 37) convenit; eademque ratione etiam reliquae formulae 37) inveniuntur.

Invento $\alpha\alpha$, e proportionibus assignatis fit:

$$\alpha\beta = \frac{n'(GG-n)+l'm'}{(GG-G'G')(GG-G''G'')},$$

quod cum 38) convenit, cuius reliquae formulae eadem methodo inveniri possunt.

15.

Postquam antecedentibus completam problematis resolutionem dedimus, sequentia adiungamus, quibus quaestio nostra haud parum illustratur, eaque adeo ad problema notum et tritum de indagatione axium principalium superficiei secundi ordinis revocatur.

E substitutionibus enim propositis per formulas 13) facile probantur aequationes sequentes:

$$39)\quad \begin{aligned} Ax+By+Cz &= Gas+G'bs'+G''cs'' \\ A'x+B'y+C'z &= Ga's+G'b's'+G''c's'' \\ A''x+B''y+C''z &= Ga''s+G'b''s'+G''c''s'' \\ Aw+A'w'+A''w'' &= G\alpha t+G'\alpha' u+G''\alpha'' v \\ Bw+B'w'+B''w'' &= G\beta t+G'\beta' u+G''\beta'' v \\ Cw+C'w'+C''w'' &= G\gamma t+G'\gamma' u+G''\gamma'' v, \end{aligned}$$

unde etiam per theorema §. 10:

$$40)\quad (Ax+By+Cz)^2+(A'x+B'y+C'z)^2+(A''x+B''y+C''z)^2 = GGss+G'G's's'+G''G''s''s'',$$

$$(Aw+A'w'+A''w'')^2+(Bw+B'w'+B''w'')^2+(Cw+C'w'+C''w'')^2 = GGtt+G'G'uu+G''G''vv,$$

quas aequationes etiam ita repraesentare licet:

$$41)\quad lxx+myy+nzz+2l'yz+2m'zx+2n'xy = GGss+G'G's's'+G''G''s''s''$$

$$pww+p'w'w'+p''w''w''+2qw'w''+2q'w''w+2q''ww' = GGtt+G'G'uu+G''G''vv.$$

Quoties vero substitutiones propositae praeter aequationes

$$\begin{aligned} xx+yy+zz &= ss+s's'+s''s'' \\ ww+w'w'+w''w'' &= tt+uu+vv, \end{aligned}$$

etiam aequationibus 40) sive 41) satisfacere debent, substitutiones illae, sicuti quantitates GG, $G'G'$, $G''G''$ determinatae sunt. Quod cum idem sit, ac si proponeretur, ellipsoidas, quae ad axes coordinatarum orthogonales relatae per aequationes exprimuntur:

$$(Ax+By+Cz)^2+(A'x+B'y+C'z)^2+(A''x+B''y+C''z)^2 = KK$$

$$(Aw+A'w'+A''w'')^2+(Bw+B'w'+B''w'')^2+(Cw+C'w'+C''w'')^2 = KK,$$

ad axes principales referre, problema ad indagationem axium principalium binarum ellipsoidarum revocatum est, quae in utraque ellipsoida fiunt $\frac{K}{G}$, $\frac{K}{G'}$, $\frac{K}{G''}$, et quarum situ substitutiones adhibendae indicantur.

Quo melius natura et situs ellipsoidarum perspiciatur, adnotemus, alterius tria puncta esse, quorum coordinatae respective sint, posito brevitatis causa $K=1$,

$$\text{primi:}\quad \frac{B'C''-B''C'}{\Delta},\quad \frac{C'A''-C''A'}{\Delta},\quad \frac{A'B''-A''B'}{\Delta},$$

$$\text{secundi:}\quad \frac{B''C-BC''}{\Delta},\quad \frac{C''A-CA''}{\Delta},\quad \frac{A''B-AB''}{\Delta},$$

tertii: $\frac{BC'-B'C}{\Delta}$, $\frac{CA'-C'A}{\Delta}$, $\frac{AB'-A'B}{\Delta}$;

iisque terminari diametros tres inter se coniugatas, quas patet perpendiculares esse tribus planis, quae aequationibus definiuntur:

$$Ax+By+Cz=0,\quad A'x+B'y+C'z=0,\quad A''x+B''y+C''z=0;$$

alterius superficiei tria puncta assignari posse, quorum coordinatae sunt,

primi: $\frac{B'C''-B''C'}{\Delta}$, $\frac{B''C-BC''}{\Delta}$, $\frac{BC'-B'C}{\Delta}$,

secundi: $\frac{C'A''-C''A'}{\Delta}$, $\frac{C''A-CA''}{\Delta}$, $\frac{CA'-C'A}{\Delta}$,

tertii: $\frac{A'B''-A''B'}{\Delta}$, $\frac{A''B-AB''}{\Delta}$, $\frac{AB'-A'B}{\Delta}$,

quibus punctis terminantur diametri superficiei tres inter se coniugatae, quas patet perpendiculares esse planis:

$$Aw+A'w'+A''w''=0,\quad Bw+B'w'+B''w''=0,\quad Cw+C'w'+C''w''=0.$$

Unde adeo problema revocatum est ad indagationem axium principalium superficierum, quarum diametri tres inter se coniugatae dantur.

Simili modo vel etiam e 40) — 41) per theorema §i 13. probantur aequationes:

$$42)\quad \begin{aligned} &[(B'C''-B''C')x+(C'A''-C''A')y+(A'B''-A''B')z]^2+\\ &[(B''C-BC'')x+(C''A-CA'')y+(A''B-AB'')z]^2+\\ &[(BC'-B'C)x+(CA'-C'A)y+(AB'-A'B)z]^2\\ &\quad = G'G'G''G''ss+G''G''GGs's'+GGG'G's''s'', \end{aligned}$$

$$\begin{aligned} &[(B'C''-B''C')w+(B''C-BC'')w'+(BC'-B'C)w'']^2+\\ &[(C'A''-C''A')w+(C''A-CA'')w'+(CA'-C'A)w'']^2+\\ &[(A'B''-A''B')w+(A''B-AB'')w'+(AB'-A'B)w'']^2\\ &\quad = G'G'G''G''tt+G''G''GGuu+GGG'G'vv, \end{aligned}$$

quibus et ipsis aliarum binarum ellipsoidarum continetur reductio ad axes earum principales.

Iam transeamus ad problema initio propositum de transformatione duplicis integralis, cuius solutionem sine calculo de quaestionibus antecedentibus deducimus.

(Cont. seq. prox.)